

MILNOR NUMBERS OF DEFORMATIONS OF SEMI-QUASI-HOMOGENEOUS PLANE CURVE SINGULARITIES II

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ABSTRACT. We show the possible Milnor numbers of deformations of semi-quasi-homogeneous isolated plane curve singularities. In Theorem 1.1 we list integers can be attained as Milnor numbers of a given semi-quasi-homogeneous singularity.

1. INTRODUCTION

Our main goal is to identify all possible Milnor numbers attained by deformations of plane curve singularities. First note that since [GZ93] it is known that not every integer less than the Milnor number of an isolated singularity f has to be a Milnor number of a deformation of f . The sequence (or possible sequences of specializations of multiparameter deformations) of Milnor numbers attained by deformations gives interesting topological data for plane curves via adjacency of μ -constant strata. Our interest in the subject stems from papers [Pło14], [Bod07] or [GLS07], as well as classic [Arn04].

The approach presented here is a continuation and fuller use of methods of [MW14] hence this paper expands the results of [MW14] and omits the assumption of irreducibility. Throughout this paper we will consider semi-quasi-homogeneous singularities (SQH for short) i.e. isolated singularities such that their initial term (in weighted Taylor expansion) is a weighted homogeneous isolated singularity. In particular, every semi-quasi-homogeneous singularity can be written in the form

$$(1) \quad f = \sum_{q\alpha + p\beta \geq pq} c_{\alpha\beta} x^\alpha y^\beta$$

for some positive integers p, q such that the initial term in $f = \sum_{q\alpha + p\beta = pq} c_{\alpha\beta} x^\alpha y^\beta$ is an isolated singularity. In such a case we will say that f as weighs $(1/p, 1/q)$ i.e. the weighs of the initial quasi-homogeneous term of f .

Without loss of generality throughout this paper we assume that

$$2 \leq p \leq q \text{ and denote } q = kp + r, p > r \geq 0.$$

We investigate the sequence of all Milnor numbers that can be attained via one-parameter deformations of f . We show that

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Theorem 1.1. *For semi-quasi-homogeneous singularity f with weights $(1/p, 1/q)$*

- (1) *if p divides q , all Milnor numbers less than $\mu(f)$ are attained except for at most:
the number $\mu(f) - (2p - 1)$ for p even
AND
numbers between $\mu(f)$ and $\mu(f) - (p - 1)$*
- (2) *if p does not divide q , all Milnor numbers less than $\mu(f)$ are attained except for at most:
the numbers between $\mu(f)$ and $\mu(f) - m$
AND
the numbers between $\mu(p, kp)$ and $\mu(p, kp) - (p - 1)$
AND
the number $\mu(p, kp) - (2p - 1)$ if p even
AND
the number $\mu(p, q) - p$ if p even and $q \equiv p - 1 \pmod{p}$*

Moreover, all these numbers are attained by linear deformations of f .

This article is organised as follows. In Section 2 we recall some standard definitions, introduce useful notation and remark on Euclid's Algorithm. Section 3 presents some steps toward the proof of Theorem 1.1. In Section 4 we present the combinatorial variant of the main Theorem 1.1 and its proof.

As a closing remark, it is important to note that in our main goal of listing integers attained as Milnor numbers of deformations we succeed only as far as non-degenerate deformations go. For degenerate deformations we are unable to present a systematic approach. Some integers missing in the list of Theorem 1.1 can be attained by degenerate deformations, but their occurrence still seems irregular and hard to present clearly. In particular, one may check that the clever approach of Brzostowski and Krasinski [BK14] does work for some singularities nicely. All suggestions or comments are welcome.

2. PRELIMINARIES

2.1. Newton numbers and diagrams. Every singularity f has a Newton diagram which here will mean the finite set of segments that give the boundary of $\text{supp} f + \mathbb{R}_+^2$ (except the two half-lines). Every such chain will be called a Newton diagram without referring to a concrete singularity.

We will say that $\nu(D)$ is the Newton number of the diagram D when it is equal to the Milnor number of a nondegenerate isolated singularity with the diagram D (see [Kou76] for the classic equivalent combinatorial definition). Note that if both end-points of two diagrams are the same, the difference between the Newton numbers of the diagrams is equal to twice the area between the diagrams.

For a diagram D we will say that the diagram \tilde{D} is its deformation if \tilde{D} arises as the convex hull of $D \cup P$, where P is some set of points in the non-negative quadrant of the lattice \mathbb{Z}^2 . Since in such a case $\nu(D) \geq \nu(\tilde{D})$, we will also write $D \geq \tilde{D}$.

For a diagram D we will say that a Newton number is attainable if it is attained by some deformation \tilde{D} of D .

2.2. Some notation. Let us introduce a convenient notation for diagrams. If $P = (p, 0)$, $Q = (0, q)$ then any translation of the segment PQ will be denoted as $\Delta(p, q)$, in other words

$$\Delta(p, q) := \text{hypotenuse of a right triangle with base of length } p \text{ and height } q$$

We will write $n\Delta(p, q)$ instead of $\Delta(np, nq)$. Moreover, for $\Delta(p_1, q_1), \dots, \Delta(p_l, q_l)$ denote by

$$(-1)^s (\Delta(p_1, q_1) + \dots + \Delta(p_l, q_l))$$

any translation of a polygonal chain with endpoints $Q, Q + (-1)^s [p_1, -q_1], \dots, Q + (-1)^s [\sum_{i=1}^l p_i, -\sum_{i=1}^l q_i]$.

2.3. Relations between singularities and the Newton diagrams. Consider a semi-quasi-homogeneous singularity f of the form (1). In particular, SQH singularity is nondegenerate. Hence, by the classic result [Kou76], its Milnor number is equal the Newton number of its diagram. Moreover, note that the weights $(1/p, 1/q)$ of a SQH singularity f such that the Newton diagram of f is contained in $\Delta(p, q)$ with both end-points on the axes are unique.

Proposition 2.1. *The only non-convenient SQH singularities are in the case $q = kp$ i.e. if $q \not\equiv 0 \pmod{p}$ then the diagram of f is of the form $\Delta(p, q)$ with end-points on both axes.*

Proof. If $p \leq q$, then the monomial y divides f if and only if the initial term of f is homogeneous (i.e. weights are $(1, 1)$). Whereas x divides f if and only if $q \pmod{p} \equiv 0$. (Indeed, it suffices to remember that the initial term of a SQH singularity has to be an isolated singularity.) \square

Note that this means that if f divisible by y , then this is the homogeneous non-convenient case and the non-degenerate deformations from [BKW14] apply. The only non-convenient case left is $q = kp$, $k > 1$. But the Milnor number of such a singularity is equal to the Milnor number of a convenient SQH singularity with p, q the same as the original one and the deformations of Lemma 4.1 apply.

Proposition 2.2. *All Newton numbers attained by deformations of Newton diagram of a SQH singularity f are equivalent to Milnor numbers attained by nondegenerate linear deformations of f .*

Proof. Let a nondegenerate singularity f have the Newton diagram D . Let \tilde{f} be the family of all nondegenerate singularities with diagram \tilde{D} . If \tilde{D} is a deformation of D , then for a generic choice of coefficients $f + t\tilde{f}$ is a deformation of f . (Here generic can mean outside an algebraic set in the jet space J^q of analytic functions in two variables that cuts transversally the Zariski closure of the family \tilde{f} .) \square

2.4. Auxiliary EEA sequence. As in [MW14] we will use a sequence arising from Extended Euclid's Algorithm. This sequence lies at the heart of the combinatorial method of constructing deformations with a given Milnor number, especially if the difference compared to $\mu(f)$ is small.

Recall that for any a, b coprime and $\epsilon = \pm 1$ there exist unique positive integers a', b' such that $a' < a, b' < b$ and $ab' - ba' = \epsilon$. Hence

Remark 2.3. [MW14, Properties 2.7 and 2.8] For a_0, b_0 coprime positive integers, there exists a finite sequence of non-negative integers $(a_j, b_j)_{j=1, \dots, l}$ such that

$$a_{j-1}b_j - b_{j-1}a_j = (-1)^{l-1-j},$$

$b_j < b_{j-1}$ for $j = 1, \dots, l$ and $a_l = 0, b_l = 1$. Additionally, we may assume $\frac{b_0}{b_1} \geq 2$ and with this condition the sequence is unique.

This sequence can be easily computed from Extended Euclid's Algorithm.

For the sequence as above denote $\text{sign}(a_j, b_j) = (-1)^{l-1-j}$. Note that signs in the sequence alternate and $\text{sign}(a_{l-1}, b_{l-1}) = 1$.

3. LEMMAS

3.1. Initial jumps for $(p, q) = m < p$. Let a_0, b_0 be coprime and consider the sequence (a_j, b_j) satisfying the conditions of Remark 2.3. We have $a_0 = Na_1 + na_2$ and $b_0 = Nb_1 + nb_2$ for some unique positive integers n, N .

Let us recall that for such n and N we have

Fact 3.1. [MW14, Propositions 3.10 and 3.13] *Let a_0, b_0 be coprime. There are deformations of a diagram $\Delta(a_0, b_0)$ that give opening terms of the sequence of minimal jumps of Newton numbers*

$$\underbrace{1, \dots, 1}_{nN}$$

Moreover, retaining the notation we have

Fact 3.2. [MW14, Remarks 3.8 and 3.14] *Let a_0, b_0 be coprime.*

If $a_1 \neq 1$, all points giving the deformations of Fact 3.1 span the diagram

$$\Sigma_{a_0, b_0} := \text{sign}(a_0, b_0) \left(N\Delta(a_1, b_1) + n\Delta(a_2, b_2) \right).$$

If $a_1 = 1$, all points giving deformations of Fact 3.1 span the diagram

$$r\Delta(1, k+1) + (a_0 - r)\Delta(1, k),$$

where $b_0 = ka_0 + r$.

Note that in particular it follows that the deformation giving the smallest Newton number (in Fact 3.1) has exactly the diagram as in Fact 3.2.

Using the above

Lemma 3.3. *If $(p, q) = m < p \leq q$, then the opening terms of the sequence of jumps of Newton numbers for a diagram $\Delta(p, q)$ are*

$$m, \underbrace{1, \dots, 1}_{r(p-r)-m}$$

Proof. For p, q coprime the Proposition is true due to [MW14, Thm 1.1]. Assume p, q are not coprime i.e. $m > 1$. We have $p = m\tilde{p}, q = m\tilde{q}$. Put $a_0 = \tilde{p}$ and $b_0 = \tilde{q}$ and consider the unique sequence $(a_j, b_j)_{j=1, \dots, l}$ from Remark 2.3.

Take the one-point deformation with the diagram

$$(2) \quad \text{sign}(a_0, b_0) (\Delta(p - a_1, q - b_1) + \Delta(a_1, b_1)).$$

This deformation gives jump m and no other deformation gives a greater Newton number.

Since $p - a_1$ and $q - b_1$ are coprime, use Fact 3.1 for the segment $\Delta(p - a_1, q - b_1)$ of the diagram (2). We can do this because due to Remark 3.2 all deformations will lie between diagram (2) and the diagram

$$\text{sign}(a_0, b_0) (\Sigma_{p-a_1, q-b_1} + \Delta(a_1, b_1)) = \text{sign}(a_1, b_1) (mN_1 \Delta(a_1, b_1) + mn_1 \Delta(a_2, b_2)),$$

which is convex. Here n_1, N_1 are natural numbers such that $b_0 = N_1 b_1 + n_1 b_2$. Hence the deformations are indeed deformations of the initial diagram $\Delta(p, q)$. Moreover, since the end-points remain fixed, the differences in Newton numbers are preserved.

Inductively, apply Fact 3.1 to segments $\Delta(mN_i a_i + a_{i+1}, mN_i b_i + b_{i+1})$ in the diagrams

$$D_i = -\text{sign}(a_i, b_i) (\Delta(mN_i a_i + a_{i+1}, mN_i b_i + b_{i+1}) + (mn_i - 1) \Delta(a_{i+1}, b_{i+1})),$$

where $N_i b_i + n_i b_{i+1} = b_0$. From Fact 3.2 we get that deformations of each D_i lie on or above the diagram

$$E_i = -\text{sign}(a_i, b_i) (\Sigma_{mN_i a_i + a_{i+1}, mN_i b_i + b_{i+1}} + (mn_i - 1) \Delta(a_{i+1}, b_{i+1})),$$

which gives the smallest Newton number. In fact

$$E_i = \text{sign}(a_i, b_i) (mN_{i+1} \Delta(a_{i+1}, b_{i+1}) + mn_{i+1} \Delta(a_{i+2}, b_{i+2}))$$

and $D_{i+1} \geq E_i$. This gives the inequality $v(D_{i+1}) \geq v(E_i)$ between Newton numbers. Therefore between every inductive step there is no gap greater than one.

Note that the last deformation due to Remark 2.3 and the second part of Fact 3.2 will have the diagram of the form

$$L = r \Delta(1, k+1) + (p-r) \Delta(1, k).$$

The above is easy to check since end-points remain fixed. Hence in the sequence of Newton numbers of deformations there is a sequence of consecutive numbers from $v(\Delta(p, q)) - m$ to $v(L) = v(\Delta(p, q)) - r(p-r)$. This ends the proof. \square

It is convenient to underline the fact below.

Remark 3.4. The Newton diagram of the sum of supports of deformations used in above Proposition 3.3 is equal

$$r \Delta(1, k+1) + (p-r) \Delta(1, k)$$

where $q = kp + r$. In particular, this is the diagram of the deformation giving the smallest Newton number on the list.

3.2. Initial jumps for mp, mq even longer $r(p-1)$.

Lemma 3.5. *If $q \equiv p-1 \pmod{p}$, we have $(p, q-1) \leq 2$ and*

$$(p, q-1) > 1 \iff p \text{ is even.}$$

Proof. Let $p = dp_0$, $q-1 = dq_0$. Obviously $d < p$. We have $q-1 = (k+1)p-2$, hence $dq_0 = d(k+1)p_0-2$. Therefore, either $d = 2$ and $q_0 - (k+1)p_0 = 1$ or $d = 1$ and $q_0 - (k+1)p_0 = 2$. The first case can occur if and only if p is even, the second if and only if p is odd (otherwise both p and $q-1$ are even and $d = 2$). \square

Lemma 3.6. *Consider $4 < p \leq q$. Denote $m = (p, q)$, $q = kp + r$, $0 < r < p$. For the Newton diagram with end-points $(0, q), (p, 0)$ all Newton numbers between $v(\Delta(p, q))$ and $v(\Delta(p, kp))$ are attained except:*

numbers between $v(\Delta(p, q))$ and $v(\Delta(p, q)) - m$ when p is odd or $q \not\equiv p-1 \pmod{p}$

or

the number $v(\Delta(p, q)) - p$ when p is even AND $q \equiv p-1 \pmod{p}$.

Proof. If $r = 0$, the theorem is trivial. If $r = 1$, then p, q are coprime and use Lemma 3.3 to get the claim.

Let us assume $q \not\equiv \pm 1$. We will consider deformations of diagrams

$$\Delta(p, q), \Delta(p, q-1), \dots, \Delta(p, q-(r-1))$$

with both end-points on the axes.

For any $l = 0, \dots, r-1$ we have

$$(3) \quad v(\Delta(p, q-l)) = v(\Delta(p, q)) - l(p-1)$$

and $q-l \equiv r-l > 0$, hence p does not divide $q-l$. Denote $m_l = (p, q-l)$. From Lemma 3.3 all numbers from $v(\Delta(p, q-l))$ to $v(\Delta(p, q-l)) - (r-l)(p-(r-l))$ are attained except for those between $v(\Delta(p, q-l))$ and $v(\Delta(p, q-l)) - m_l$.

We will make sure that these sequences give consecutively the sequence of jumps equal 1, i.e. that the missing Newton numbers in each step l are already covered by Newton numbers in the preceding step.

First, we show that $v(p, q-l) - m_l \geq v(p, q-(l+1))$. Indeed, since $m_l < p$, we get $-l(p-1) \geq -(l+1)(p-1) + m_l$ and from equality (3) we get the claim.

Now it suffices to show that the first attained number after the gap in step $l+1$ is bigger than the last number attained in step l i.e.

$$(4) \quad v(\Delta(p, q-(l+1))) - m_{l+1} \geq v(\Delta(p, q-l)) - (r-l)(p-(r-l))$$

for $l = 0, \dots, r-2$. Indeed, note that $2 \leq r-l \leq p-2$ and therefore

$$\min_{l=0, \dots, r-2} (r-l)(p-(r-l)) \geq 2(p-2).$$

Since $m_{l+1} < p$ and it divides p , we get $p-3 \geq m_{l+1}$ for $p \geq 5$. Hence

$$(r-l)(p-(r-l)) - (p-1) \geq 2(p-2) - (p-1) = p-3 \geq m_{l+1}.$$

Combine with the fact $v(\Delta(p, q-(l+1))) = v(\Delta(p, q-l)) - (p-1)$ and we get inequality (4).

Therefore we get that all numbers between $v(\Delta(p, q)) - m$ and $v(\Delta(p, q - (r - 1))) - (p - 1) = v(\Delta(p, q)) - r(p - 1)$ are attained which gives the claim of the lemma.

Now assume that $q \equiv p - 1$. In particular, p, q are coprime and Lemma 3.3 shows that all numbers from $v(\Delta(p, q))$ to $v(\Delta(p, q)) - (p - 1)$ are attained. For $p, q - 1$ we have $q - 1 \equiv p - 2 \not\equiv p - 1$, hence we can apply the reasoning above. Note that $v(\Delta(p, q - 1)) = v(\Delta(p, q)) - (p - 1)$ and $(p, q - 1) \leq 2$ depending on whether p is even or odd due to Lemma 3.5, hence all numbers from $v(\Delta(p, q))$ to $v(\Delta(p, q)) - (p - 1)(p - 2) - (p - 1)$ are attained except for $v(\Delta(p, q)) - p$ if p is even. This ends the proof. \square

4. MAIN RESULTS COMBINATORIALLY

Theorem 4.1. *Consider p and $q = kp$. For any Newton diagram contained in the diagram $\Delta(p, kp)$ with end-points $(0, kp), (p, 0)$ all numbers are attained except numbers between $v(\Delta(p, kp))$ and $v(\Delta(p, kp)) - (p - 1)$ and the number $v(\Delta(p, q)) - (2p - 1)$ when p is even.*

Proof. Assume $p > 4, k > 1$ and that the diagram is equal $\Delta(p, kp)$ with end-points on both axes.

For any $\kappa \in \mathbb{N}$ first jump for $\Delta(p, \kappa p)$ is $p - 1$ attained by deformation $\Delta(p, \kappa p - 1)$. Since $\kappa p - 1 \equiv p - 1 \pmod{p}$, we use Lemma 3.6 and get all numbers from $v(\Delta(p, \kappa p - 1)) = v(\Delta(p, \kappa p)) - (p - 1)$ to $v(\Delta(p, \kappa p - 1)) - (p - 1)^2 = v(\Delta(p, \kappa p)) - p(p - 1)$ except $v(\Delta(p, \kappa p - 1)) - p = v(\Delta(p, \kappa p)) - (2p - 1)$ when p even. Note that $v(\Delta(p, \kappa p)) - p(p - 1) = v(\Delta(p, (\kappa - 1)p))$ for $\kappa \geq 2$.

Consider diagrams

$$(5) \quad \Delta(2, p + 2\kappa) + \Delta(i - 2, (i - 2)\kappa) + \Delta(p - i, \kappa(p - i) - 1),$$

for $i = 2, \dots, p - 1$ with end-points $(p, 0)$ and $(0, (\kappa + 1)p - 1)$. For $\kappa < k$ they are deformations of $\Delta(p, kp)$.

For $\kappa \geq 2$ the above deformations (5) give all numbers between $v(\Delta(p, \kappa p))$ and $v(\Delta(p, \kappa p)) - (p - 1)$. If $\kappa = 1$ then above deformations give all numbers between $v(\Delta(p, p))$ and $v(\Delta(p, p)) - (p - 1) + 1 = v(\Delta(p, p - 1)) + 2$. The number $v(\Delta(p, p - 1)) + 1$ is attained by the deformation $\Delta(2, p + 6) + \Delta(p - 2, p - 4)$ of $\Delta(p, kp)$.

Moreover, one can check that the deformation $v(\Delta(2, 2\kappa + 3)) + \Delta(p - 2, (p - 2)\kappa - 2)$ gives the number $v(p, \kappa p) - (2p - 1)$ for $\kappa < k$.

Hence by induction for any $k \geq 2$ we get that all numbers from $v(\Delta(p, kp))$ to 1 are attained except for:

numbers between $v(\Delta(p, kp))$ and $v(\Delta(p, kp)) - (p - 1)$

AND

$v(\Delta(p, kp)) - (2p - 1)$ if p is an even number.

For $p = 1$ the germ f is smooth. Now note that for $p = 2, 3, 4$ the claim holds, see Lemma 4.3 below. To conclude the proof, note that the Newton number of a Newton diagram contained in $\Delta(p, kp)$ with end-points on both axes has the same Newton number as the bigger diagram. Moreover, all deformations in the considerations above are deformations also of such a smaller diagram. Moreover, for

$k = 1$ the claim holds also, see [BKW14, Thm BKW nzdeg] and Remark 4.2 below. This ends the proof. \square

Remark 4.2. Consider $\Delta(p, p)$, $4 \geq p \geq 2$. All numbers between $\nu(p, p) - (p - 1)$ and 1 are attained. Indeed, the first non-degenerate jump is equal to $p - 1$. One can check by hand that all numbers from $\nu(\Delta(p, p - 1)) = \nu(\Delta(p, p)) - (p - 1)$ to 1 are attained.

The following Lemma can be proven to great extent by methods of proof of Theorem 4.1. Nevertheless, we thought it might be instructive to provide explicit deformations in the case of small p .

Lemma 4.3. Consider a diagram $\Delta(p, kp)$ for $p = 2, 3, 4$ and $k \geq 2$. The claim of Theorem 4.1 holds.

Proof. We will show explicitly the deformations needed depending on p . Take any positive integer $\kappa \leq k$.

Consider $p = 2$. For $\Delta(2, 2k)$ by deformations $\Delta(2, 2k - i)$, $i = 1, \dots, 2k - 2$ we get all numbers from $\nu(\Delta(2, 2k))$ to 1.

Consider $p = 3$. The first jump for $\Delta(3, 3k)$ is 2 attained by deformation $\Delta(3, 3k - 1)$ and

- deformations $i\Delta(1, \kappa) + \Delta(3 - i, \kappa(3 - i) - 1)$, $i = 0, 1, 2$ give numbers from $\nu(\Delta(3, 3\kappa - 1))$ to $\nu(\Delta(3, 3\kappa - 1)) - 2$ for $\kappa \geq 2$
- the deformation $\Delta(2, 2\kappa - 1) + \Delta(1, \kappa - 1)$ gives the number $\nu(\Delta(3, 3\kappa - 1)) - 3$ for $\kappa \geq 2$
- the deformation $\Delta(3, 3\kappa - 3)$ gives the number $\nu(\Delta(3, 3\kappa - 1)) - 4$ for $\kappa \geq 2$
- deformations $\Delta(2, 2\kappa + 1) + \Delta(1, \kappa - 2)$ for $\kappa \geq 2$ and $\Delta(2, 4)$ for $\kappa = 2$ give the number $\nu(\Delta(3, 3\kappa - 1)) - 5$

Hence by Remark 4.2 we get that for $p = 3$ all numbers from $\nu(\Delta(p, kp))$ to 1 are attained except for $\nu(\Delta(3, 3k)) - 1$.

Consider $p = 4$. The first jump for $\Delta(4, 4k)$ is 3 attained by deformation $\Delta(4, 4k - 1)$ and

- deformations $i\Delta(1, \kappa) + \Delta(4 - i, (4 - i)\kappa - 1)$ for $i = 0, \dots, 3$ give numbers from $\nu(\Delta(4, 4\kappa - 1))$ to $\nu(\Delta(4, 4\kappa - 1)) - 3$ for $\kappa \geq 2$.
 - deformations $i\Delta(1, \kappa) + \Delta(3 - i, (3 - i)\kappa - 1) + \Delta(1, \kappa - 1)$ for $i = 0, 1, 2$ give numbers $\nu(\Delta(4, 4\kappa - 1)) - 5$, $\nu(\Delta(4, 4\kappa - 1)) - 6$, $\nu(\Delta(4, 4\kappa - 1)) - 7$ for $\kappa \geq 2$.
 - deformations $i\Delta(1, \kappa) + \Delta(2 - i, (2 - i)\kappa - 1) + \Delta(2, 2\kappa - 2)$ for $i = 0, 1$ give numbers $\nu(\Delta(4, 4\kappa - 1)) - 8$, $\nu(\Delta(4, 4\kappa - 1)) - 9$ for $\kappa \geq 2$. Note that $\nu(\Delta(4, 4(\kappa - 1))) = \nu(\Delta(4, 4\kappa - 1)) - 9$.
 - deformations $\Delta(2, 2\kappa + 3 - i) + \Delta(2, 2\kappa - 3)$ for $i = 1, 2$ give numbers $\nu(\Delta(4, 4(\kappa - 1))) - 1$ and $\nu(\Delta(4, 4(\kappa - 1))) - 2$ for $\kappa \geq 2$.
 - deformations $\Delta(2, 2\kappa + 1) + \Delta(2, 2\kappa - 4)$ for $\kappa > 2$ and $\Delta(2, 3)$ for $\kappa = 2$ give the number $\nu(\Delta(4, 4(\kappa - 1))) - 7$.
- (Note that the deformation $\Delta(3, 8)$ gives the number $\nu(\Delta(4, 4k)) - 7$ for $k = 2$ but for $k > 2$ there is no deformation which gives this number.)

Hence by Remark 4.2 we get that for $p = 4$ all numbers from $v(\Delta(p, kp))$ to 1 are attained except for $v(\Delta(4, 4k)) - 1$, $v(\Delta(4, 4k)) - 2$ and $v(\Delta(4, 4k)) - 7$ (the last gap disappears when $k = 2$). \square

Theorem 4.4. *Consider $4 < p < q$. Denote $q = kp + r$, $0 < r < p$ and $m = (p, q)$. All numbers from $v(\Delta(p, q))$ to 1 are attained as Newton numbers except for at most:*

numbers between $v(\Delta(p, q))$ and $v(\Delta(p, q)) - m$

and

numbers between $v(p, kp)$ and $v(\Delta(p, kp)) - (p - 1)$

and

the number $v(p, q) - p$ if p even and $r = p - 1$

and

the number $v(\Delta(p, kp)) - (2p - 1)$ if p is even.

Proof. Combine Lemma 3.6 and Theorem 4.1 to get the claim. \square

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